MATB41 Weeks 11 - 13 Notes

1. Triple Integral in a Rectangular Box:

- Let $B = [a,b] \times [c,d] \times [p,q]$ be a **compact** (Bounded and Closed) box in $\mathbb{R}^3$.

- Let $f: B \to \mathbb{R}$ be a cont function. Proceeding as in double integrals, we can partition the 3 sides of B into n equal parts and form the Riemann Sum:

$$\sum_{i,j,k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta V$$

- The limit of the Riemann Sum, if it exists, is the **triple integral of f over B.**

$$\iiint_B f(x,y,z)\, dV = \lim_{n \to \infty} \sum_{i,j,k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta V$$

- Let f be integrable on the box $B = [a,b] \times [c,d] \times [p,q]$. The triple integral of f over B may be evaluated by any of its iterated integrals.

I.e. $$\iiint_B f(x,y,z)\, dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\, dz\, dy\, dx$$

$$= \int_a^b \int_p^q \int_c^d f(x,y,z)\, dy\, dz\, dx$$

$$= \int_c^d \int_a^b \int_p^q f(x,y,z)\, dz\, dx\, dy$$

$$= \int_c^d \int_p^q \int_a^b f(x,y,z)\, dx\, dz\, dy$$

$$= \int_p^q \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

$$= \int_p^q \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz$$

- E.g. Find the volume of the box b given by $f(x,y,z) = z-2$ where $b = \{(x,y,z) \mid 0 \le x \le 3, 0 \le y \le 2, 0 \le z \le 1\}$.

Soln:

$$\iiint_B f(x,y,z)\,dV = \int_0^1 \int_0^2 \int_0^3 z-2 \,dx\,dy\,dz$$
$$= \int_0^1 (z-2)(6)\,dz$$
$$= 6\int_0^1 z-2\,dz$$
$$= 6\left[\frac{z^2}{2} - 2z \Big|_0^1\right]$$
$$= 6\left[\frac{1}{2} - 2\right]$$
$$= 6\left(-\frac{3}{2}\right)$$
$$= -9$$

$$\int_0^3 \int_0^1 \int_0^2 z-2\,dy\,dz\,dx$$
$$= \int_0^3 \int_0^1 2(z-2)\,dz\,dx$$
$$= 2\int_0^3 \left(-\frac{3}{2}\right) dx$$
$$= \frac{(-6)}{2} \int_0^3 dx$$
$$= -9, \text{ as before}$$

Note: Like with double integrals, always choose the easiest and simplest way to calculate the answer, unless otherwise specified.

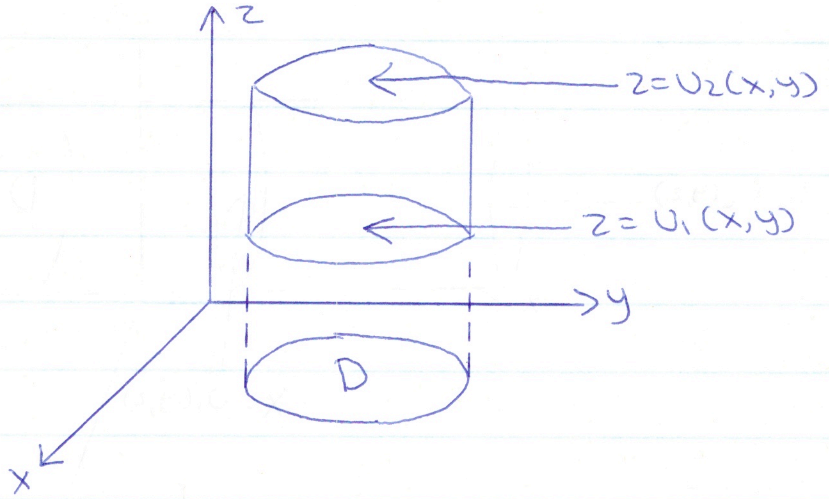
- The volume of the three-dimension region E is given by:

$$V = \iiint_E dV.$$

- Elementary Regions:
  - An elementary region in 3-D space is defined by restricting one of the variables to be between 2 functions of the remaining functions.
    - This is for general regions.
    - There are 3 types of elementary regions.

Type 1:

Let W be a region in R^3.



$W = \{(x,y,z) \mid (x,y) \in D,\ U_1(x,y) \le z \le U_2(x,y)\}$

$U_1(x,y)$ and $U_2(x,y)$ are cont in D.

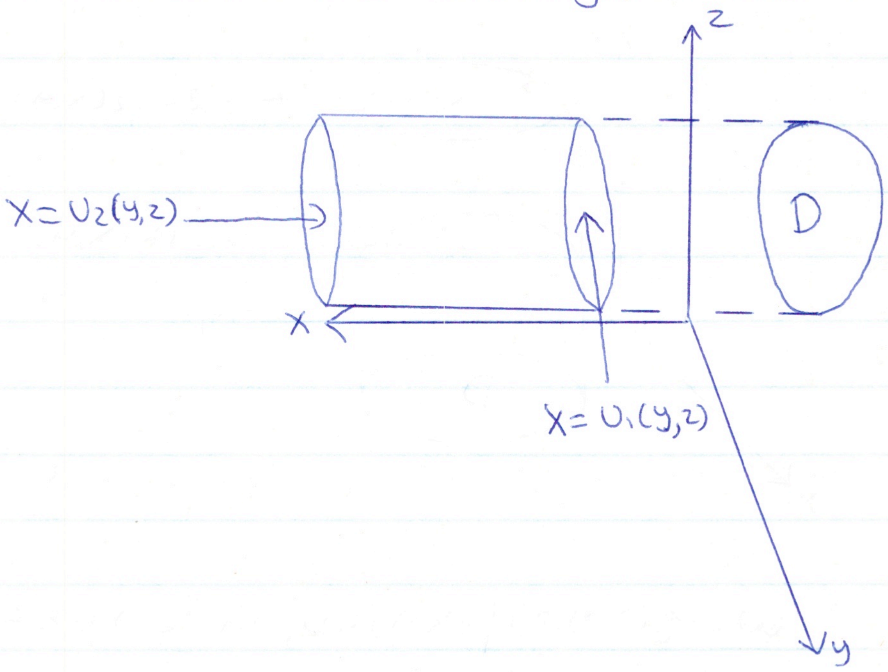
$(x,y) \in D$ means that (x,y) lies in the region D from the xy-plane.

It is either x-simple or y-simple.

$$\iiint_W f(x,y,z)\,dv = \iint_D \left(\int_{U_1(x,y)}^{U_2(x,y)} f(x,y,z)\,dz \right) dA$$

Type 2:

Let ω be a region in R^3.



$W = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

$u_1(y,z)$ and $u_2(y,z)$ are cont in D.

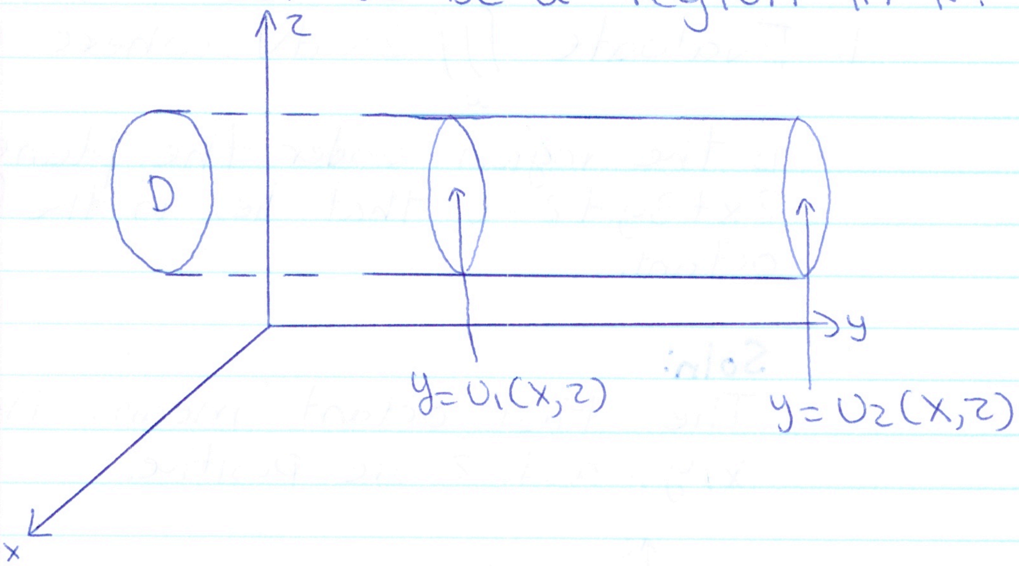
$(y,z) \in D$ means that (y,z) lies in the region D from the yz-plane.

It is either y-simple or z-simple.

$$\iiint_{\omega} f(x,y,z)\,dv = \iint_{D} \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

Type 3:

Let ω be a region in R^3.



$W = \{(x,y,z) \mid (x,z) \in D,\ U_1(x,z) \leq y \leq U_2(x,z)\}$

$U_1(x,z)$ and $U_2(x,z)$ are cont in D.

$(x,z) \in D$ means that (x,z) lies in the region D from the xz-plane.

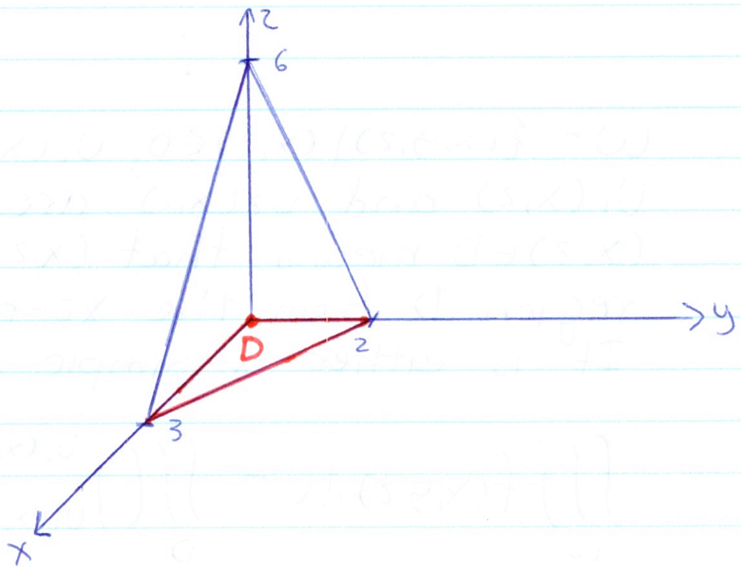
It is either x-simple or z-simple.

$$\iiint_{\omega} f(x,y,z)\,dv = \iint_{D} \left(\int_{U_1(x,z)}^{U_2(x,z)} f(x,y,z)\,dy \right) dA$$

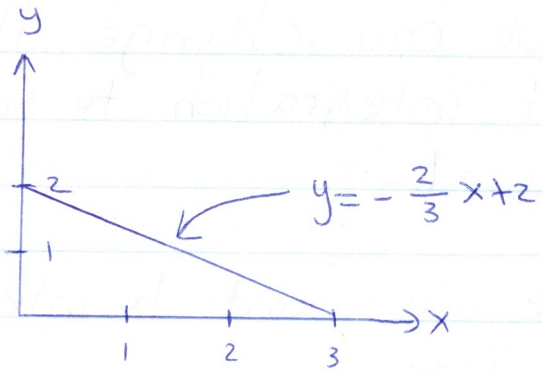
- Examples:

1. Evaluate $\iiint_W 2x\, dV$ where W is the region under the plane $2x+3y+z=6$ that lies in the first octant.

Soln:
The first octant means that x, y, and z are positive.



From the diagram, we see that D on the xy-plane is a triangle bounded by the points $(0,0)$, $(3,0)$ and $(0,2)$.



Furthermore, we know that $0 \le z \le 6 - 2x - 3y$, $0 \le x \le 3$, and $0 \le y \le -\frac{2}{3}x + 2$.

We can integrate this using Type 1.

$$\iint_D \left(\int_0^{6-2x-3y} 2x \, dz \right) dA$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} 2x \, dz\, dy\, dx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} (2x)(6-2x-3y) \, dy\, dx$$

$$= \int_0^3 \frac{4}{3}x^3 - 8x^2 + 12 \, dx$$

$$= 9$$

Note, we can change the order of integration to solve the integral.

Suppose we want to integrate using Type 2.

I.e. $\iiint f(x,y,z)\,dx\,dy\,dz$.

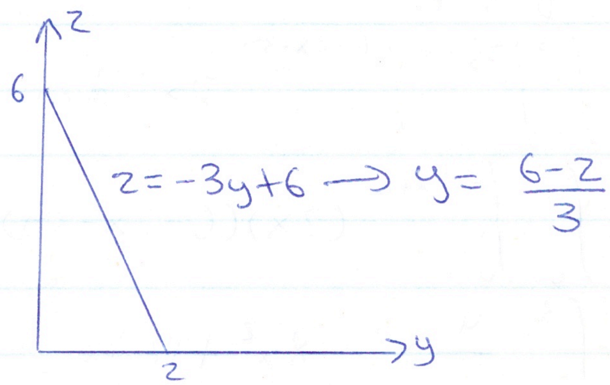
Soln:

$2x + 3y + z = 6$

$2x = 6 - 3y - z$

$x = 3 - \frac{3}{2}y - \frac{z}{2}$

$0 \le x \le 3 - \frac{3}{2}y - \frac{z}{2}$



$0 \le y \le \frac{6-z}{3}$

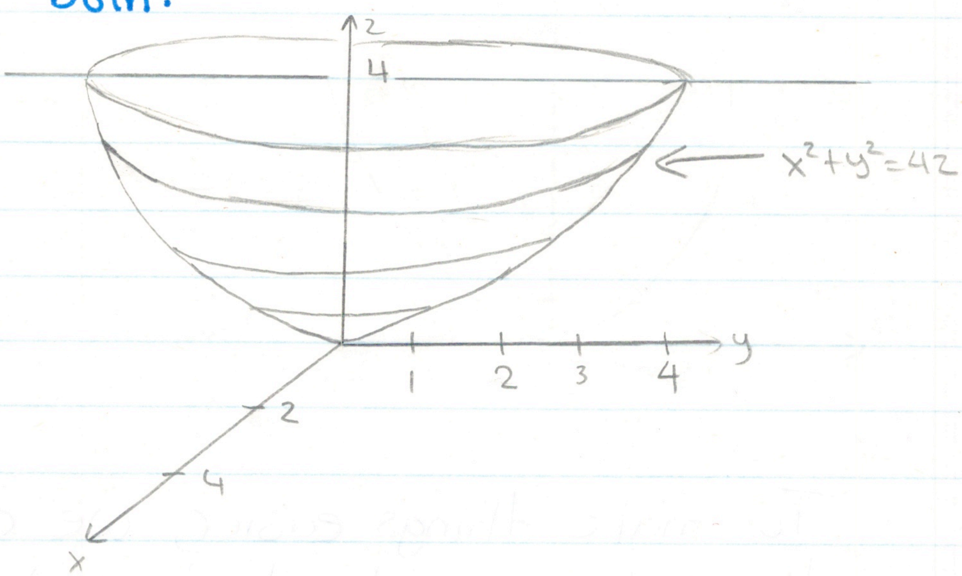
$0 \le z \le 6$

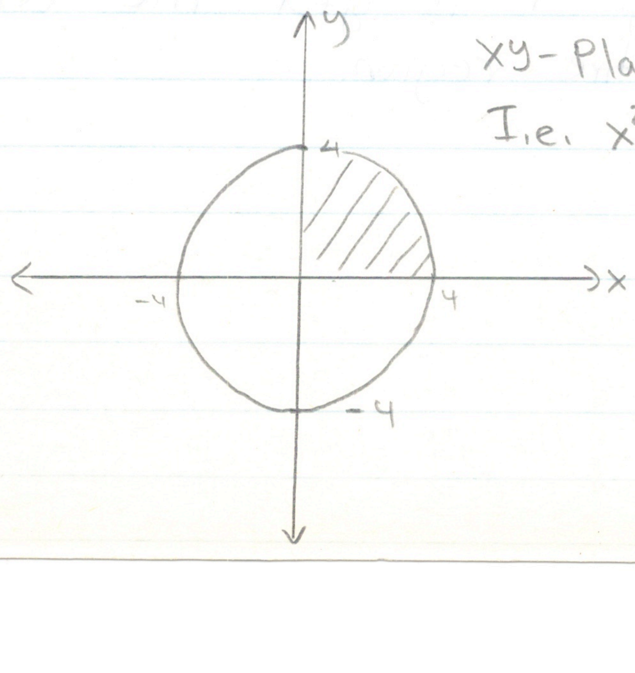
$$\int_0^6 \int_0^{\frac{6-z}{3}} \int_0^{3-\frac{3y}{2}-\frac{z}{2}} 2x \, dx\,dy\,dz = 9$$

2. Changing the Order of Integration:

- E.g. Express the triple integral for the vol bounded by $4z = x^2 + y^2$ and $z = 4$ in
  1. $dz\,dy\,dx$
  2. $dy\,dx\,dz$
  3. $dx\,dz\,dy$

Soln:

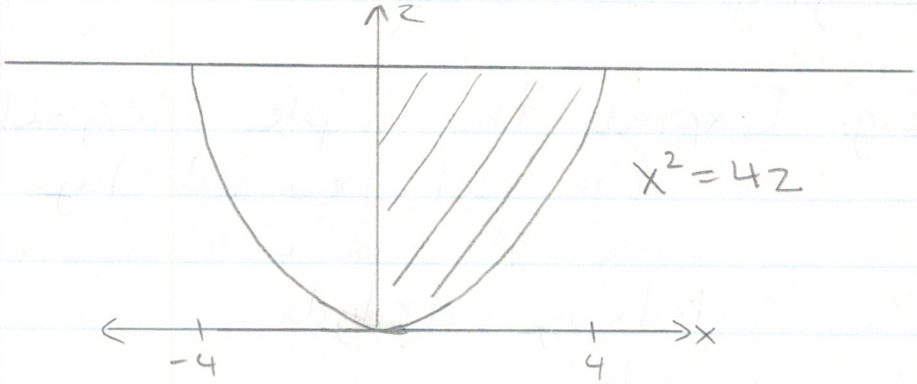




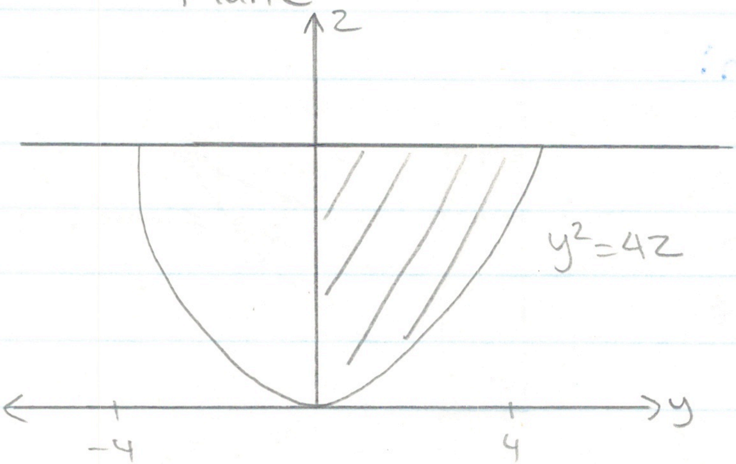
xy-Plane, when $z=4$

I.e. $x^2+y^2=16$

XZ- Plane



YZ- Plane



To make things easier, we can calculate the volume in 1 octant and multiply that by 4 to get the volume of the entire region.

1. dzdydx

$$V = 4\int_0^4 \int_0^{\sqrt{16-x^2}} \int_{\frac{x^2+y^2}{4}}^{4} dz\,dy\,dx$$

We want z to be in terms of x and/or y. We have the formula $4z = x^2+y^2$. Rearranging that gives us $z = \frac{x^2+y^2}{4}$. This is the lower bound. The upper bound is $z=4$.

Now, we want y in terms of x. We have the equation $x^2+y^2=16$. This becomes $y=\sqrt{16-x^2}$. If you look at the picture of $x^2+y^2=16$, and think of it in terms of y-simple, you'll see that $0 \le y \le \sqrt{16-x^2}$ and $0 \le x \le 4$.

2. dy dx dz

$$V = 4\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$$

This time, we want y in terms of x and/or z. $y=\sqrt{4z-x^2}$. This is the upper bound. The lower bound is $y=0$.

Now, we have dxdz. We can treat this as x-simple on the xz-plane. $x^2=4z \rightarrow x=2\sqrt{z}$. Looking at the picture of the xz-plane, we see $0 \le x \le 2\sqrt{z}$ and $0 \le z \le 4$.

3. dx dz dy

$$V=4\int_0^4 \int_{\frac{y^2}{4}}^{4} \int_0^{\sqrt{4z-x^2}} dx\,dz\,dy$$

We want x in terms of y and z. $x=\sqrt{4z-y^2}$. This is the upper bound. The lower bound is $x=0$.

We can treat dzdy as z-simple on the yz-plane. $\frac{y^2}{4} \le z \le 4$ and

$0 \le y \le 4$.

- Tips / Tricks:
  1. Draw out the 3-D graph and all the 2-D graphs.
  2. Rewrite the formulas in terms of the specified variable.
  3. Once you're done with the innermost variable, treat the other variables as x-simple, y-simple or z-simple.

3. Change of Variables for Double Integrals:

- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a diff map. Let $D = \text{dom}(T) \subset uv$-plane and range$(T) \subset xy$-plane.

$$T(u,v) = (x(u,v), y(u,v))$$

- A transformation T from the uv-plane to the xy-plane is just a map taking a point in the uv-plane into a point in the xy-plane.

- T is 1-1 on D if $T(u_1, v_1) = T(u_2, v_2)$ implies that $u_1 = u_2$ and $v_1 = v_2$.

- If T is 1-1, then T has an inverse T^{-1} from the xy-plane to the uv-plane.

- Let $T: D \subset R^2 \to R^2$ be a C^1 transformation given by $T(u,v) = (x(u,v), y(u,v))$. The **Jacobian Determinant** of T, written as $\frac{\partial(x,y)}{\partial(u,v)}$ is the det of the derivative matrix $D T(u,v)$ of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- Let $T: D^* \subset R^2 \to D \subset R^2$ be a C^1 transformation given by $T(u,v) = (x(u,v), y(u,v))$.

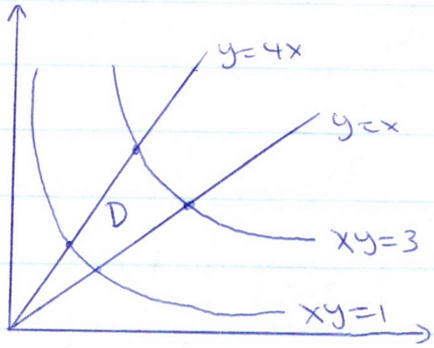
abs value

$$A(D^*) = \iint_{D^*} dA^* \qquad A(D) = \iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

– E.g. Find the area of D where D is bounded by:

1. $y = x$
2. $y = 4x$
3. $xy = 1$
4. $xy = 3$

Soln:



$y = 4x \longrightarrow \frac{y}{x} = 4$, $y = x \longrightarrow \frac{y}{x} = 1$ } Let $u = \frac{y}{x}$, $1 \leq u \leq 4$

$xy = 1$, $xy = 3$ } Let $v = xy$, $1 \leq v \leq xy$

$$uv = \left(\frac{y}{x}\right)(xy) = y^2$$

$$y = \sqrt{uv}$$

$$\frac{v}{u} = \frac{xy}{\left(\frac{y}{x}\right)} = x^2 \Rightarrow x = \sqrt{\frac{v}{u}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{2} v^{\frac{1}{2}} u^{-\frac{3}{2}} & \frac{1}{2} v^{-\frac{1}{2}} u^{-\frac{1}{2}} \\ \frac{1}{2} v^{\frac{1}{2}} u^{-\frac{1}{2}} & \frac{1}{2} v^{-\frac{1}{2}} u^{\frac{1}{2}} \end{vmatrix}$$

$$= \frac{-1}{2u}$$

$$\iint_D dA = \iint_{D^*} \left| \frac{-1}{2u} \right| du\,dv$$

$$= \int_1^3 \int_1^4 \left| \frac{-1}{2u} \right| du\,dv$$

$$= \frac{1}{2} \int_1^3 \int_1^4 \frac{1}{u} \, du\,dv$$

$$= \frac{1}{2} \int_1^3 \ln(4) \, dv$$

$$= \ln(4)$$

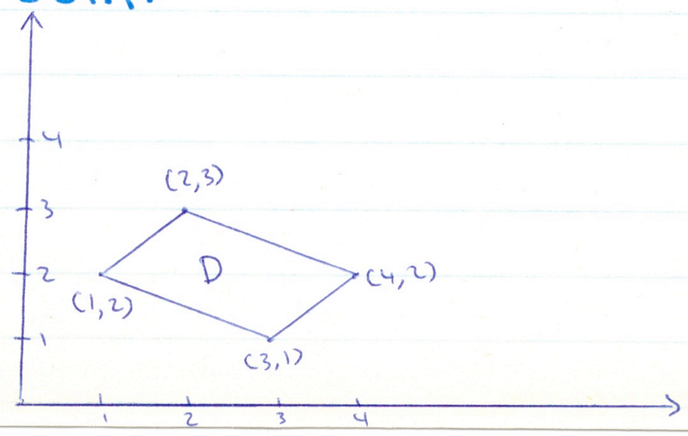
– Thm: Let D and D^* be elementary regions, x-simple or y-simple, in $\mathbb{R}^2$ and let $T: D^* \to D$ be a 1-1 C^1 transformation given by $T(u,v) = (x(u,v), y(u,v))$ with $D = T(D^*)$. Then, for any integrable function $f: D \to \mathbb{R}$, we have:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

abs value $\longrightarrow$

– E.g. Evaluate $\iint_D 2y^2 - x^2 - xy\,dx\,dy$ where D is the region enclosed by the 4 points $(1,2)$, $(2,3)$, $(3,1)$ and $(4,2)$.

Soln:



The eqn of the line connecting (1,2) and (3,1) can be represented by

$$\frac{y-3}{x-2} = \frac{3-2}{2-4}$$

$$= \frac{-1}{2}$$

$$y-3 = (-\tfrac{1}{2})(x-2)$$

$$2y-6 = -x+2$$

$$x+2y = 8$$

The eqn of the line connecting (2,3) and (4,2) can be represented by

$$\frac{y-2}{x-1} = \frac{2-1}{1-3}$$

$$= -\frac{1}{2}$$

$$2y-4 = -x+1$$

$$x+2y = 5$$

Let $u = x+2y$.

$5 \leq u \leq 8$.

The eqn of the line connecting (2,3) and (1,2) can be represented by

$$\frac{y-3}{x-2} = \frac{3-2}{2-1}$$

$$y-3 = x-2$$

$$y-x = 1$$

$$x-y = -1$$

The eqn of the line connecting (3,1) and (4,2) can be represented by

$$\frac{y-2}{x-4} = \frac{2-1}{4-3}$$

$$y-2 = x-4$$

$$y-x = -2$$

$$x-y = 2$$

Let $v = x-y$.

$$-1 \leq v \leq 2$$

$$v = x-y$$

$$u = x+2y$$

$2v+u = 3x$ $\qquad$ $u-v = 3y$

$x = \frac{1}{3}(u+2v)$ $\qquad$ $y = \frac{1}{3}(u-v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$
$$= -\frac{1}{3}$$

$$f(x(u,v), y(u,v)) = 2\left(\tfrac{1}{3}(u-v)\right)^2 - \left(\tfrac{1}{3}(u+2v)\right)^2 - \left(\tfrac{1}{3}(u-v)\right)\left(\tfrac{1}{3}(u+2v)\right)$$
$$= -uv$$
$$\iint_D (2y^2-x^2-xy)\,dx\,dy = \int_5^8 \int_{-1}^{2} (-uv)\left|-\tfrac{1}{3}\right| du\,dv$$
$$= \frac{-39}{4}$$

4. Double Integrals in Polar Coordinates:

- A type of change of variable.
- Recall:
  $x = r\cos\theta$
  $y = r\sin\theta$

The Jacobian Matrix is
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r$$

- Let f be cont on the region in the xy-plane.
  Let $R = \{(r,\theta) \mid 0 \le h_1(\theta) \le r \le h_2(\theta),$
  $\alpha \le \theta \le \beta\}$, where $0 \le \beta - \alpha \le 2\pi$.
  Then:
  $$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(r,\theta), y(r,\theta))\,|r|\,dr\,d\theta$$

- E.g. Find the vol of the region beneath the surface $z = xy + 10$ and above the annular region $D = \{(x,y) \mid 4 \le x^2+y^2 \le 16\}$ on the xy-plane.

Soln:

Let $x = r\cos\theta$

Let $y = r\sin\theta$

$x^2+y^2 = r^2$

$4 \le r^2 \le 16 \longrightarrow 2 \le r \le 4$

$0 \le \theta \le 2\pi$

$$\iint_D z\,dA = \int_0^{2\pi}\int_2^4 (r^2\sin\theta\cos\theta + 10)\,|r|\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^4 \left(\frac{r^2}{2} 2\sin\theta\cos\theta + 10\right)|r|\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^4 \left(\frac{r^2}{2}\sin(2\theta) + 10\right)|r|\,dr\,d\theta$$

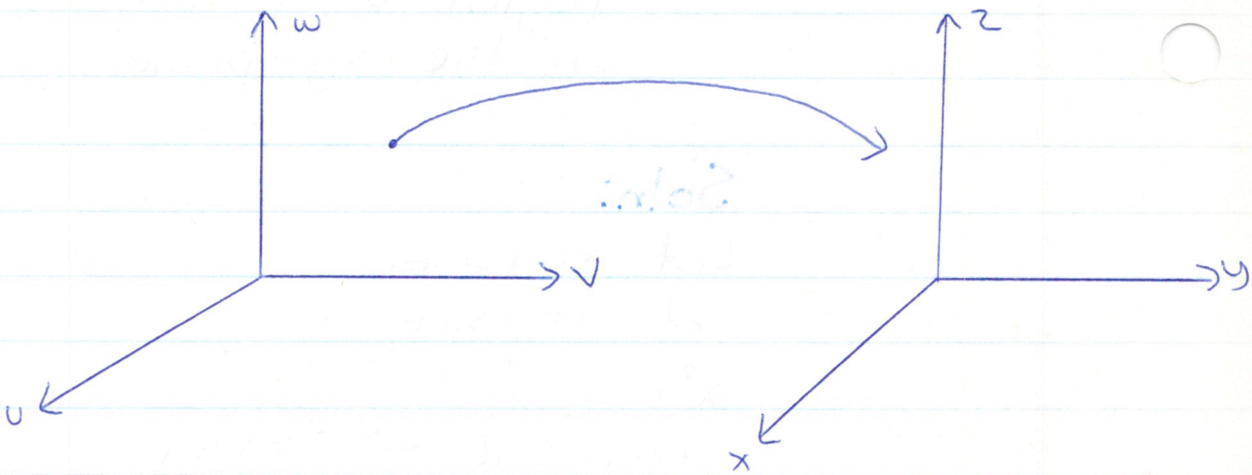
$$= \frac{1}{2}\int_0^{2\pi}\int_2^4 \left(r^3\sin(2\theta) + 10r\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{r^4}{4}(\sin(2\theta)) + 5r^2 \Big|_2^4 \right] d\theta$$

$$= 120\pi$$

5. Change of Variables in Triple Integration:

- T is a transformation from R^3 (u,v,w space) to R^3 (x,y,z space). Let T map a region S in uvw-space to a region R in xyz-space.



$$T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w)).$$

- Let $T: S \subset \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 transf given by $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$. The Jacobian det of T, denoted by $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is the det of the derivative matrix $DT(u,v,w)$ of T:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Thm: Let W and W^* be elementary regions in $\mathbb{R}^3$ and let $T: W^* \to W$ be a 1-1 C^1 transf given by $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ with $W = T(W^*)$. Then, for any integrable function $f: W \to \mathbb{R}$, we have

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

abs value $\longrightarrow$

- E.g. Eval $\iiint_{\omega} xz\,dv$ where ω is a parallelepiped bounded by the planes
  1. $y=x$
  2. $y=x+2$
  3. $z=x$
  4. $z=x+3$
  5. $z=0$
  6. $z=4$

Soln:

$$\left.\begin{array}{l} y=x \rightarrow y-x=0 \\ y=x+2 \rightarrow y-x=2 \end{array}\right\} \begin{array}{l} \text{Let } u=y-x \\ 0\le u\le 2 \end{array}$$

$$\left.\begin{array}{l} z=x \rightarrow z-x=0 \\ z=x+3 \rightarrow z-x=3 \end{array}\right\} \begin{array}{l} \text{Let } v=z-x \\ 0\le v\le 3 \end{array}$$

$$\left.\begin{array}{l} z=0 \\ z=4 \end{array}\right\} \begin{array}{l} \text{Let } w=z \\ 0\le w\le 4 \end{array}$$

$$\omega^* = \{(u,v,w) \mid 0\le u\le 2,\ 0\le v\le 3,\ 0\le w\le 4\}$$

$$\begin{array}{ll} v = z-x & u = y-x \\ \quad = w-x & \quad = y+(-x) \\ -x = v-w & \quad = y+v-w \\ x = w-v & y = u+w-v \end{array}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

$$\iiint_{\omega} xz\,dv = \iiint_{\omega^*} (w-v)(w)\,|1|\,du\,dv\,dw$$

$$= \int_0^4 \int_0^3 \int_0^2 w^2 - wv \; du\,dv\,dw$$

$$= 56$$

6. Triple Integrals in Cylindrical Coordinates:

- A type of change of variable.

- Recall:

  $x = r\cos\theta$

  $y = r\sin\theta$

  $z = z$

  $x^2 + y^2 = r^2$

- The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\,(r)\,dr\,d\theta\,dz$$

7. Triple Integrals in Spherical Coordinates:

- A type of change of variable.

- Recall:

$$(x,y,z) \longrightarrow (\rho, \theta, \phi)$$

$$\left.\begin{aligned} r &= \rho\sin\phi \\ x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho &= \sqrt{x^2+y^2+z^2} \end{aligned}\right\} \quad \begin{aligned} &0 \le \phi \le \pi \\ &0 \le \theta \le 2\pi \\ &0 \le \rho < +\infty \end{aligned}$$

- The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\phi & \rho\sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= -\rho^2\sin\phi$$

- $$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$